ON THE ABSOLUTELY CONTINUOUS SPECTRUM OF DIRAC OPERATOR

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Abstract. We prove that the massless Dirac operator in $\mathbb{R}^3$ with long-range potential has an a.c. spectrum which fills the whole real line. The Dirac operators with matrix-valued potentials are considered as well.

1. Introduction

In this paper, we consider Dirac operator for the massless particle in the external field generated by the long-range potential

$$H = -i\alpha \cdot \bigtriangledown + V \tag{1}$$

Here

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Matrices σ_j are called the Pauli matrices. Denote the exterior of the unit ball in $\mathbb{R}^3$ by Ω. We use a notation Σ for the unit sphere in $\mathbb{R}^3$. Consider H in the Hilbert space $[L^2(\Omega)]^4$. Assume that the elements of a self-adjoint 4×4 matrix-function $V(x)$ are uniformly bounded in Ω. We also assume that V has certain canonical form after the spherical change of variables. There are many meaningful potentials that satisfy these assumptions [20]. Consider the self-adjoint operator $\mathcal{H}$, generated by the boundary conditions $f_3(x) = f_4(x) = 0$ as $x \in \Sigma$. We introduce a matrix

$$\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The main result of the paper is the following statement.

Theorem 3.1. *Assume $V(x) = v(x)\beta$, where $v(x)$ is a real-valued, uniformly bounded, scalar function satisfying the following condition*

$$\int\limits_{x \in \Omega} \frac{v^2(x)}{|x|^2 + 1} < \infty \tag{2}$$

Then, $\sigma_{ac}(\mathcal{H}) = \mathbb{R}$.

This result can be regarded as a PDE (Partial Differential Equations) analog of the celebrated results by Szegö on polynomials orthogonal on the unit circle with ℓ^2 Verblunsky parameters [17, 19]. If one considers the power-decaying potentials $v(x) : |v(x)| \leq C(|x| + 1)^{-0.5-\varepsilon}$, then condition (2) is satisfied for any $\varepsilon > 0$. In this case, we also obtain an asymptotics for the Green's function.

For the massless Dirac operator, theorem 3.1 solves Simon's conjecture [18] for Schrödinger operators. Under more conditions on V, the spectrum of $\mathcal{H}$ is purely a.c. on $\mathbb{R}$ [22]. One can easily construct an example when conditions of the theorem 3.1 are satisfied and the rich singular spectrum occurs. In the one-dimensional case, the first result on the presence of a.c. spectrum for slowly decaying potentials is due to M. Krein [9]. See also [4, 12, 6] for the modern development. The existence of wave operators for the one-dimensional Dirac operator with square summable potential was proved in [7]. An interesting paper [10] discusses Szegö-type inequalities for Schrödinger operators with short-range potentials. We will use some ideas from [10]. For discrete multidimensional Schrödinger operator with random slow decay, the existence of wave operators was proved by Bourgain [3].

The structure of the paper is as follows. In the second section, we consider one-dimensional Dirac systems with matrix-valued potentials. Then, in the third part, we deal with a multidimensional Dirac operator.

The following notations will be used. $|M| = \sqrt{M^*M}$ denotes the absolute value of matrix M, symbol 1 will often stand for the identity matrix or operator. As usual, $C_0^\infty(\mathbb{R}^+)$ denotes the space of infinitely smooth functions (or vector-functions) with the compact support inside $(0,\infty)$. $\chi_K(x)$ denotes the characteristic function of the set K. $\langle x, y\rangle$ stands for the inner product of two vectors x and y in $\mathbb{R}^3$.

2. One-dimensional Dirac operator with matrix-valued potential

In this section, we study the one-dimensional case. Let us consider the Dirac operator in the following form

$$D = \begin{bmatrix} 0 & -d/dr \\ d/dr & 0 \end{bmatrix} + V, \quad V = \begin{bmatrix} -b & -a \\ -a & b \end{bmatrix} \tag{3}$$

where $a(r), b(r)$ are $m \times m$ self-adjoint matrices with locally integrable entries. This form of a general Dirac operator is called a canonical form ([11], pp.48–50). The elements of the Hilbert space are $f = (f_1, f_2)^t$ with $f_{1(2)} \in \left[L^2(\mathbb{R}^+)\right]^m$. The boundary condition $f_2(0) = 0$ defines the self-adjoint operator $\mathcal{D}$. We start with an elementary spectral theory of $\mathcal{D}$. Consider solutions of the following equation

$$D \begin{bmatrix} \Phi \\ \Psi \end{bmatrix} = \lambda \begin{bmatrix} \Phi \\ \Psi \end{bmatrix}, \Phi(0,\lambda) = 1, \Psi(0,\lambda) = 0, \lambda \in \mathbb{C}$$

For any f, consider the generalized Fourier transform

$$F(\lambda) = \int_0^\infty \Phi^*(r,\lambda) f_1(r)dr + \int_0^\infty \Psi^*(r,\lambda) f_2(r) dr$$

There exists non-decreasing $m \times m$ matrix-function $\sigma(\lambda)$, $\lambda \in \mathbb{R}$ (spectral matrix-valued measure) such that ([14], p.106)

$$\int_0^\infty \left(|f_1(r)|^2 + |f_2(r)|^2\right) dr = \int_{-\infty}^\infty F^*(\lambda) d\sigma(\lambda) F(\lambda) \tag{4}$$

$$\int_{-\infty}^\infty \frac{d\sigma(\lambda)}{1+\lambda^2} < \infty \tag{5}$$

The resolvent kernel $R_z(r,s)$ of operator $\mathcal{D}$ has the following form

$$R_z(r,s)=\int\limits_{-\infty}^{\infty}\frac{1}{\lambda-z}\begin{bmatrix}\Phi(r,\lambda)d\sigma(\lambda)\Phi^*(s,\lambda) & \Phi(r,\lambda)d\sigma(\lambda)\Psi^*(s,\lambda)\\ \Psi(r,\lambda)d\sigma(\lambda)\Phi^*(s,\lambda) & \Psi(r,\lambda)d\sigma(\lambda)\Psi^*(s,\lambda)\end{bmatrix} \tag{6}$$

where the integral is understood in the distributional sense. Notice that

$$\operatorname{Im} R_z(0,0)=\begin{bmatrix}\operatorname{Im}\int\limits_{-\infty}^{\infty}(\lambda-z)^{-1}d\sigma(\lambda) & 0\\ 0 & 0\end{bmatrix} \tag{7}$$

where the integral converges due to (5). Assume $a(r)=b(r)=0$, if $r>R$. Then, we can always find the Jost solution $F(r,\lambda)$:

$$DF=\lambda F,\ F(r,\lambda)=\begin{bmatrix}F_1(r,\lambda)\\ F_2(r,\lambda)\end{bmatrix}=e^{i\lambda r}\begin{bmatrix}-i\\ 1\end{bmatrix},\ \text{for } r>R, \lambda\in\overline{\mathbb{C}^+}$$

Notice that $F_2(0,\lambda)$ is an entire matrix-valued function. It is non-degenerate in $\mathbb{C}^+$. Indeed, otherwise we would have the non-real eigenvalue for the self-adjoint operator $\mathcal{D}$. The matrix

$$\begin{bmatrix}\Phi(r,\lambda) & F_1(r,\lambda)\\ \Psi(r,\lambda) & F_2(r,\lambda)\end{bmatrix}$$

is non-degenerate for $r=0,\lambda\in\mathbb{C}^+$, therefore, it is non-degenerate for any $r>0$. Consider the following matrix

$$Z(r,\lambda)=\begin{bmatrix}\Phi(r,\lambda) & F_1(r,\lambda)\\ \Psi(r,\lambda) & F_2(r,\lambda)\end{bmatrix}^{-1}=\begin{bmatrix}Z_{11}(r,\lambda) & Z_{12}(r,\lambda)\\ Z_{21}(r,\lambda) & Z_{22}(r,\lambda)\end{bmatrix}$$

Then, the resolvent of $\mathcal{D}$ can be written in the following form

$$R_\lambda\begin{bmatrix}h_1\\ h_2\end{bmatrix}=\begin{bmatrix}\Phi(r,\lambda)\\ \Psi(r,\lambda)\end{bmatrix}\int\limits_r^{\infty}\left(-Z_{11}(s,\lambda)h_2(s)+Z_{12}(s,\lambda)h_1(s)\right)ds+$$

$$\begin{bmatrix}F_1(r,\lambda)\\ F_2(r,\lambda)\end{bmatrix}\int\limits_0^{r}\left(Z_{21}(s,\lambda)h_2(s)-Z_{22}(s,\lambda)h_1(s)\right)ds,\lambda\in\mathbb{C}^+$$

Consequently, the resolvent kernel can be written in the form

$$R_\lambda(r,s)=\begin{cases}\begin{bmatrix}\Phi(r,\lambda)Z_{12}(s,\lambda) & -\Phi(r,\lambda)Z_{11}(s,\lambda)\\ \Psi(r,\lambda)Z_{12}(s,\lambda) & -\Psi(r,\lambda)Z_{11}(s,\lambda)\end{bmatrix}, & \text{if } r<s\\[2ex] \begin{bmatrix}-F_1(r,\lambda)Z_{22}(s,\lambda) & F_1(r,\lambda)Z_{21}(s,\lambda)\\ -F_2(r,\lambda)Z_{22}(s,\lambda) & F_2(r,\lambda)Z_{21}(s,\lambda)\end{bmatrix}, & \text{if } r>s\end{cases}$$

In the free case ($a=b=0$), $\Phi(r,\lambda)=\cos(\lambda r)$, $\Psi(r,\lambda)=-\sin(\lambda r)$, $F_1(r,\lambda)=-i\exp(i\lambda r)$, $F_2(r,\lambda)=\exp(i\lambda r)$, $\sigma(\lambda)=d\lambda/\pi$, and

$$R^0_\lambda(r,s)=\begin{cases}\begin{bmatrix}i\cos(\lambda r)\exp(i\lambda s) & -\cos(\lambda r)\exp(i\lambda s)\\ -i\sin(\lambda r)\exp(i\lambda s) & \sin(\lambda r)\exp(i\lambda s)\end{bmatrix}, & \text{if } r<s\\[2ex] \begin{bmatrix}i\exp(i\lambda r)\cos(\lambda s) & -i\exp(i\lambda r)\sin(\lambda s)\\ -\exp(i\lambda r)\cos(\lambda s) & \exp(i\lambda r)\sin(\lambda s)\end{bmatrix}, & \text{if } r>s\end{cases}$$

Notice that $Z(0,\lambda)=\left[\begin{array}{cc}1 & -F_1(0,\lambda)F_2^{-1}(0,\lambda)\\ 0 & F_2^{-1}(0,\lambda)\end{array}\right]$ and

$$\operatorname{Im} R_\lambda(0,0)=\frac{1}{2i}\left[R_\lambda(0,0)-R_\lambda^*(0,0)\right]=$$

$$=\frac{1}{2i}\left[\begin{array}{cc}F_2^{*-1}(0,\lambda)F_1^*(0,\lambda)-F_1(0,\lambda)F_2^{-1}(0,\lambda) & 0\\ 0 & 0\end{array}\right],\ \lambda\in\mathbb{C}^+ \tag{8}$$

For $\lambda\in\mathbb{C}$, let us introduce the following functions: $f_1(r,\lambda)=\exp(-i\lambda r)F_1(r,\lambda)$, $f_2(r,\lambda)=\exp(-i\lambda r)F_2(r,\lambda)$. Functions f_1 and f_2 satisfy the following equations

$$\left\{\begin{array}{l} f_1'=-i\lambda f_1+af_1+\lambda f_2-bf_2\\ f_2'=-\lambda f_1-bf_1-i\lambda f_2-af_2\end{array}\right.$$

Therefore, a simple algebra yields

$$\frac{d}{dr}\left(\frac{f_1^*f_2-f_2^*f_1}{i}\right)=-2\operatorname{Im}\lambda\,|f_1+if_2|^2 \tag{9}$$

Using the asymptotics at infinity, we obtain

$$\frac{f_1^*(r,\lambda)f_2(r,\lambda)-f_2^*(r,\lambda)f_1(r,\lambda)}{i}=2+2\operatorname{Im}\lambda\int\limits_r^\infty|f_1(\rho,\lambda)+if_2(\rho,\lambda)|^2d\rho \tag{10}$$

Lemma 2.1. *The following relations hold*

$$\frac{F_1^*(r,\lambda)F_2(r,\lambda)-F_2^*(r,\lambda)F_1(r,\lambda)}{i}=2,\lambda\in\mathbb{R} \tag{11}$$

$$\frac{F_1^*(0,\lambda)F_2(0,\lambda)-F_2^*(0,\lambda)F_1(0,\lambda)}{i}\geq 2,\lambda\in\mathbb{C}^+ \tag{12}$$

Proof. It suffices to use (10) and definition of f_1 and f_2. ■

Notice that the lemma is wrong if the Dirac operator is not in the canonical form. (11) implies that $F_2(0,\lambda)$ is non-degenerate on $\mathbb{R}$ also.

Corollary. The following relation holds: $\operatorname{Im} R_\lambda(0,0)_{(1,1)}=F_2^{*-1}(0,\lambda)F_2^{-1}(0,\lambda)$, if λ is real (see (8)). Therefore, using (7), we obtain

$$\sigma'(\lambda)=\pi^{-1}F_2^{*-1}(0,\lambda)F_2^{-1}(0,\lambda) \tag{13}$$

The factorization results of this sort are quite common in the scattering theory. Direct analogs hold for the so-called Krein systems (factorization via Π–functions [9]), for one-dimensional Schrödinger operators with matrix-valued potentials (factorization via Jost functions [1]), for polynomials with matrix-valued coefficients (factorization via Szegö function [5, 2, 19]). Notice that the Krein system with coefficient $-A(r)$ yields operator $-\mathfrak{D}$ [9].

Using the representation of resolvent by the solutions, one gets

$$R_\lambda(r,0)=\left[\begin{array}{cc}-F_1(r,\lambda)F_2^{-1}(0,\lambda) & 0\\ -F_2(r,\lambda)F_2^{-1}(0,\lambda) & 0\end{array}\right],\ \text{for } \lambda\in\overline{\mathbb{C}^+} \tag{14}$$

So, we can find $F_2^{-1}(0,\lambda)$ by following a simple rule that turns out to be applicable to PDE

$$\lim_{r\to\infty}R_\lambda(r,0)e^{-i\lambda r}=\left[\begin{array}{cc}iF_2^{-1}(0,\lambda) & 0\\ -F_2^{-1}(0,\lambda) & 0\end{array}\right],\ \lambda\in\overline{\mathbb{C}^+}$$

Lemma 2.2. *Consider $a(r), b(r) \in C_0^\infty(\mathbb{R}^+)$. Then, for the function $F_2(0,\lambda)$, the following inequalities hold true*

$$\|F_2(0,iy)\| \le \exp\left[Cy^{-1}\int_0^\infty [\|a(r)\|^2 + \|b(r)\|^2]dr\right] \tag{15}$$

The asymptotics of $F_2(0,\lambda)$ as $\lambda \in \overline{\mathbb{C}^+}, |\lambda| \to \infty$ is

$$F_2(0,\lambda) = 1 - \frac{1}{2i\lambda}\int_0^\infty |b(r) + ia(r)|^2 dr + O(|\lambda|^{-2} \tag{16}$$

Proof. Consider a matrix

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Upon multiplying by J, the equation $DY = \lambda Y$ can be rewritten as

$$Y' + QY = -\lambda JY, Q = -JV \tag{17}$$

Take

$$Y_0 = \begin{bmatrix} -ie^{i\lambda r} & e^{-i\lambda r} \\ e^{i\lambda r} & -ie^{-i\lambda r} \end{bmatrix}$$

and find the solution to (17) in the following form $Y = Y_0 S$.

Thus

$$S' = -Y_0^{-1}QY_0S = \begin{bmatrix} 0 & (-b+ia)e^{-2i\lambda r} \\ (-b-ia)e^{2i\lambda r} & 0 \end{bmatrix} S \tag{18}$$

Imposing condition

$$S(\infty,\lambda) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

on the $2m \times m$ matrix

$$S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$$

we get $F = Y_0 S$. System (18) can be rewritten in the following form

$$S_1(r,\lambda) = 1 + \int_r^\infty e^{-2i\lambda s}(b(s) - ia(s))S_2(s,\lambda)ds \tag{19}$$

$$S_2(r,\lambda) = \int_r^\infty e^{2i\lambda s}(b(s) + ia(s))S_1(s,\lambda)ds \tag{20}$$

These two equations are easy to iterate. We obtain the following integral equation for S_1

$$S_1(r,\lambda) = 1 + \int_r^\infty e^{-2i\lambda s}(b(s) - ia(s))\int_s^\infty e^{2i\lambda t}(b(t) + ia(t))S_1(t,\lambda)dtds \tag{21}$$

One needs to use Gronwall's inequality to obtain the first statement of the lemma. Iterating (21) and integrating by parts, one proves (16). ■

Remark. Notice that the usual scattering coefficients $A(\lambda)$ and $B(\lambda)$ can be calculated by the formulas

$$\begin{aligned} A(\lambda) &= \frac{1}{2}(F_2(0,\lambda) + iF_1(0,\lambda)) = S_1(0,\lambda), \\ B(\lambda) &= \frac{1}{2}(F_2(0,\lambda) - iF_1(0,\lambda)) = -iS_2(0,\lambda), \lambda \in \mathbb{C} \end{aligned} \tag{22}$$

If $a, b \in C_0^\infty(\mathbb{R}^+)$, then the asymptotics of $A(\lambda)$ is

$$A(\lambda) = 1 - \frac{1}{2i\lambda}\int\limits_0^\infty |b(r) + ia(r)|^2 dr + O(|\lambda|^{-2}), \lambda \to \infty, \lambda \in \overline{\mathbb{C}^+} \tag{23}$$

Using (11), we get the conservation law

$$|A(\lambda)|^2 = 1 + |B(\lambda)|^2, \ \lambda \in \mathbb{R} \tag{24}$$

From (12),

$$|A(\lambda)|^2 \geq 1 + |B(\lambda)|^2, \ \lambda \in \mathbb{C}^+ \tag{25}$$

Notice that if $\xi \in \mathbb{C}^m, \ \xi \neq 0$, then the function $\ln ||F_2^{-1}(0,\lambda)\xi||$ is subharmonic in $\mathbb{C}^+$. Thus, the following inequality holds

$$\pi \ln ||F_2^{-1}(0,iy)\xi|| \leq y \int\limits_{-\infty}^{\infty} \frac{\ln ||F_2^{-1}(0,\lambda)\xi||}{\lambda^2 + y^2} d\lambda = \frac{y}{2}\int\limits_{-\infty}^{\infty} \frac{\ln(\pi\sigma'(\lambda)\xi,\xi)}{\lambda^2 + y^2} d\lambda \tag{26}$$

Here we used asymptotics of $F_2^{-1}(0,\lambda)$ as $|\lambda| \to \infty$ and its continuity in $\overline{\mathbb{C}^+}$. The following theorem is straightforward.

Theorem 2.1. *Assume that* $||a(r)||, ||b(r)|| \in L^2(\mathbb{R}^+)$, $\xi \in \mathbb{C}^m$, $\|\xi\| = 1$. *Then, the following inequality holds*

$$y^2 \int\limits_{-\infty}^{\infty} \frac{\ln(\pi\sigma'(\lambda)\xi,\xi)}{y^2 + \lambda^2} d\lambda > -C \int\limits_0^\infty \left[\|a(r)\|^2 + \|b(r)\|^2\right] dr \tag{27}$$

Proof. Assume $a(r), b(r) \in \mathbb{C}_0^\infty(\mathbb{R}^+)$ first. Then, the estimate (15) from lemma 2.2 implies

$$|F_2^{-1}(0,iy)| \geq \exp\left(-Cy^{-1}\int\limits_0^\infty \left[\|a(r)\|^2 + \|b(r)\|^2\right] dr\right) \tag{28}$$

So, (26) gives the needed estimate. Now, consider any $||a(r)||, ||b(r)|| \in L^2(\mathbb{R}^+)$. We can find the sequence $a_n(r), b_n(r) \in \mathbb{C}_0^\infty(\mathbb{R}^+)$ such that

$$\int\limits_0^\infty (\|a_n - a\|^2 + \|b_n - b\|^2) dr \to 0$$

as $n \to \infty$. Let σ_n denote the spectral measure for a_n, b_n. Then, the second resolvent identity yields the weak convergence of measures σ_n to σ. For each individual σ_n, we have the estimate (27). Then, using the standard argument involving semicontinuity of the entropy [8], one gets (28). ■

Remark. Constants in the inequality (27) do not depend on m, the size of the matrix. Therefore, one can easily prove an analogous result for the Dirac operator with square summable operator-valued potential. As a simple corollary, $\sigma_{ac}(\mathcal{D}) = \mathbb{R}$.

Remark. The similar results can be proved for the Krein systems [9, 15] with matrix-valued or operator-valued coefficients. Notice that the Krein system with coefficient $-A(r)$ in a standard way generates the Dirac operator $\mathcal{D}_g = -\mathcal{D}$ with $\mathcal{D}$ introduced by (3).

Assume that $a(r), b(r)$ are integrable with finite support. Consider a fixed vector $\xi \in \mathbb{C}^m$. Let us find an element of the Hilbert space with generalized Fourier transform $(\lambda - iy)^{-1}\xi$. This function is given by the formula

$$\begin{bmatrix} \int\limits_{-\infty}^{\infty} (\lambda - iy)^{-1}\Phi(r,\lambda)d\sigma(\lambda)\xi \\ \int\limits_{-\infty}^{\infty} (\lambda - iy)^{-1}\Psi(r,\lambda)d\sigma(\lambda)\xi \end{bmatrix}$$

and is equal to $R_{iy}(r,0)[\xi,0]^t$ due to (6). At the same time, (14) says that this vector is equal to

$$-\begin{bmatrix} F_1(r,iy)F_2^{-1}(0,iy)\xi \\ F_2(r,iy)F_2^{-1}(0,iy)\xi \end{bmatrix}$$

Therefore, the following arguments are valid. If the vector-function

$$h(r) = (h_1(r), h_2(r))^t \in \mathbb{C}^{2m} \tag{29}$$

satisfies equation $Dh = iyh$ and decays at infinity, then $h_1(r) = F_1(r,iy)\eta, h_2(r) = F_2(r,iy)\eta$, where $\eta \in \mathbb{C}^m$. Take $\xi = F_2(0,iy)\eta = h_2(0)$. The generalized Fourier transform for the function $h(r)$ is $-(\lambda - iy)^{-1}h_2(0)$. The spectral measure of h is

$$\sigma(\lambda,h) = \int\limits_{-\infty}^{\lambda} \frac{d(\sigma(\lambda)h_2(0), h_2(0))}{\lambda^2 + y^2}$$

Using the estimate (26), we have

$$y\int\limits_{-\infty}^{\infty} \frac{\ln\left[\pi(\sigma'(\lambda,h)(\lambda^2+y^2))\right]}{\lambda^2+y^2}d\lambda \geq 2\pi \ln ||F_2^{-1}(0,iy)h_2(0)|| = 2\pi \ln ||\eta||$$

or

$$y\int\limits_{-\infty}^{\infty} \frac{\ln\left[(\sigma'(\lambda,h)\right]}{\lambda^2+y^2}d\lambda \geq C + 2\pi \ln ||\eta|| \tag{30}$$

where the constant C depends on y. This inequality will play the crucial role later. Roughly speaking, it means the following.

If we found at least one solution having the "free" asymptotics at some point in $\mathbb{C}^+$, then the entropy of the spectral measure can be controlled by the "amplitude", i.e. $||\eta||$.

The definition of the spectral measure gives an equality

$$\int_{-\infty}^{\infty} \sigma'(\lambda,h)d\lambda = ||h||^2$$

Therefore, the lower bound for $||\eta||$ yields the lower bound for

$$\int_{-\infty}^{\infty} \frac{\ln^- \sigma'(\lambda,h)}{\lambda^2+y^2}d\lambda \tag{31}$$

Consider $a(r), b(r) \in \mathbb{C}_0^\infty(\mathbb{R}^+)$, $||\xi||=1$. The functions $\ln|(F^{-1}(0,\lambda)\xi,\xi)|$, $\ln|(A^{-1}(\lambda)\xi,\xi)|$, $\ln\|A(\lambda)\|$ are subharmonic in $\mathbb{C}^+$ and tend to 0 as $|\lambda|\to\infty, \lambda\in\overline{\mathbb{C}^+}$. Write the corresponding inequality at $\lambda = iy$. Taking $y\to\infty$, one gets the following estimates by comparing the coefficients against y^{-1}.

$$\int_{-\infty}^{\infty} \ln(\pi\sigma'(\lambda)\xi,\xi)d\lambda = 2\int_{-\infty}^{\infty} \ln\|F_2^{-1}(0,\lambda)\xi\|d\lambda \geq$$

$$\geq 2\int_{-\infty}^{\infty} \ln|(F_2^{-1}(0,\lambda)\xi,\xi)|d\lambda > -C\int_0^\infty \left[||b(r)\xi+ia(r)\xi||^2\right]dr;$$

$$\int_{-\infty}^{\infty} \ln|(A^{-1}(\lambda)\xi,\xi)|d\lambda > -C\int_0^\infty \left[||b(r)\xi+ia(r)\xi||^2\right]dr;$$

$$\int_{-\infty}^{\infty} \ln||A^{-1}(\lambda)||d\lambda > -C\int_0^\infty [\|a(r)\|^2+\|b(r)\|^2]dr$$

Notice that estimates $||A^{-1}||\leq 1$, $||A^{*-1}||\leq 1$ imply

$$\begin{gathered}\int_\Delta \ln|(A^{-1}(\lambda)\xi,\xi)|d\lambda > -C\int_0^\infty \left[||b(r)\xi+ia(r)\xi||^2\right]dr\\ \int_\Delta \ln||A^{-1}(\lambda)||d\lambda > -C\int_0^\infty [\|a(r)\|^2+\|b(r)\|^2]dr\end{gathered} \tag{32}$$

for any $\Delta\subset\mathbb{R}$. Consider Dirac operator (3). Fix some $\xi\in\mathbb{C}^m$, $\|\xi\|=1$. Estimate (32) suggests that condition $(b(r)+ia(r))\xi\in L^2(\mathbb{R}^+)$ guarantees $\sigma_{ac}(\mathcal{D})=\mathbb{R}$.

Theorem 2.2. *Let $a(r)$ and $b(r)$ have locally summable entries. Assume that there is a fixed vector $\xi\in\mathbb{C}^m$ such that $(b+ia)\xi\in L^2(\mathbb{R}^+)$. Then, $\sigma_{ac}(\mathcal{D})=\mathbb{R}$.*

Proof. Without loss of generality, assume $\xi=e_1=(1,0,\ldots,0)^t$. Let us start with the compactly supported a and b. In this case, the Jost solution $F(r,\lambda)$ makes sense. Consider two functions $f_1(\lambda)=A(\lambda)F_2^{-1}(0,\lambda)e_1$ and $f_2(\lambda)=B(\lambda)F_2^{-1}(0,\lambda)e_1$. From (24), $\|f_1(\lambda)\|^2=\|F_2^{-1}(0,\lambda)e_1\|^2+\|f_2(\lambda)\|^2$ for real λ. Since $f_1+f_2=e_1$ (by (22)), we have $|(f_1(\lambda),e_1)|^2=\|F_2^{-1}(0,\lambda)e_1\|^2+|(f_2(\lambda),e_1)|^2\geq|(F_2^{-1}(0,\lambda)e_1,e_1)|^2+|(f_2(\lambda),e_1)|^2$. Consider

$$\alpha(\lambda)=\frac{(f_1(\lambda),e_1)}{(F_2^{-1}(0,\lambda)e_1,e_1)}, \beta(\lambda)=\frac{(f_2(\lambda),e_1)}{(F_2^{-1}(0,\lambda)e_1,e_1)}$$

Then,

$$\alpha(\lambda)+\beta(\lambda)=\frac{1}{(F_2^{-1}(0,\lambda)e_1,e_1)}$$

and

$$|\alpha(\lambda)|^2\geq 1+|\beta(\lambda)|^2, \lambda\in\mathbb{R}$$

Therefore,

$$|(F_2^{-1}(0,\lambda)e_1,e_1)|\geq \frac{1}{2|\alpha(\lambda)|}, \lambda\in\mathbb{R}$$

Function $(F_2^{-1}(0,\lambda)e_1,e_1)$ is analytic in $\overline{\mathbb{C}^+}$ and tends to 1 as $|\lambda|\to\infty$, $\lambda\in\overline{\mathbb{C}^+}$. The numerator of $\alpha(\lambda)$, function $(f_1(\lambda),e_1)$, does not have zeroes in $\overline{\mathbb{C}^+}$. Indeed, in $\overline{\mathbb{C}^+}$, we have $\|f_1(\lambda)\|^2\geq\|F_2^{-1}(0,\lambda)e_1\|^2+\|f_2(\lambda)\|^2$ (by (25)) and $f_1(\lambda)+f_2(\lambda)=e_1$. Thus, $|(f_1(\lambda),e_1)|^2\geq\|F_2^{-1}(0,\lambda)e_1\|^2+|(f_2(\lambda),e_1)|^2\geq\|F_2^{-1}(0,\lambda)e_1\|^2>0$. So, the function $\ln|\alpha(\lambda)|$ is superharmonic and nonnegative on $\mathbb{R}$. For any $\Delta\subset\mathbb{R}$, we have the estimates

$$\int\limits_\Delta \ln[\pi(\sigma'(\lambda)e_1,e_1)]d\lambda\geq 2\int\limits_\Delta \ln\|F^{-1}(0,\lambda)e_1\|d\lambda\geq 2\int\limits_\Delta \ln|(F^{-1}(0,\lambda)e_1,e_1)|d\lambda$$
$$\geq -2\int\limits_\Delta \ln|\alpha(\lambda)|d\lambda-2\,|\Delta|\,\ln 2 \tag{33}$$

Superharmonicity of $\ln|\alpha(\lambda)|$ implies

$$\pi\ln|\alpha(iy)|\geq y\int\limits_{-\infty}^{\infty}\frac{\ln|\alpha(\lambda)|}{\lambda^2+y^2}d\lambda \tag{34}$$

Consider $a_\varepsilon, b_\varepsilon\in C_0^\infty(\mathbb{R}^+)$ that approximate a,b in $L^2(\mathbb{R}^+)$ norm as $\varepsilon\to 0$. Denote the corresponding spectral measure by σ_ε. Apply (33) and (34). Take $y\to+\infty$ in (34) and use lemma 2.2 and (23). We obtain

$$\int\limits_{-\infty}^{\infty}\ln|\alpha_\varepsilon(\lambda)|d\lambda\leq\frac{\pi}{2}\int\limits_0^\infty\|(b_\varepsilon(r)+ia_\varepsilon(r))e_1\|^2dr \tag{35}$$

Estimates (33) and (35) yield

$$\int\limits_\Delta \ln(\sigma_\varepsilon'(\lambda)e_1,e_1)d\lambda\geq C_1+C_2\int\limits_0^\infty\|(b_\varepsilon(r)+ia_\varepsilon(r))e_1\|^2dr \tag{36}$$

Since $d\sigma_\varepsilon$ converges weakly to $d\sigma$, we have

$$\int\limits_\Delta \ln(\sigma'(\lambda)e_1,e_1)d\lambda\geq C_1+C_2\int\limits_0^\infty\|(b(r)+ia(r))e_1\|^2dr \tag{37}$$

Now, take arbitrary $a(r)$ and $b(r)$. Consider truncations $a_n(r)=a(r)\chi_{[0,n]}(r)$, $b_n(r)=b(r)\chi_{[0,n]}(r)$. Since the functions a_n and b_n are compactly supported, estimate (37) holds for the corresponding measure $\sigma_n(\lambda)$. It is the general fact of

the spectral theory, that σ_n converges to σ weakly as $n \to \infty$. Take $n \to \infty$ to obtain

$$\int_{\Delta} \ln(\sigma'(\lambda)e_1, e_1)d\lambda \geq C_1 + C_2 \int_0^{\infty} \|(b(r) + ia(r))e_1\|^2 dr \tag{38}$$

That completes the proof. ■

Since the constants C_1 and C_2 in (38) are independent of m, the size of the matrices, the theorem is true for the operator-valued Dirac systems as well. Its statement is very strong, it proves a certain rigidity of the Dirac operator (3)[1]. That makes theorem 2.2 applicable even to some PDE.

3. Multidimensional Dirac operator

In this section, we consider two operators

$$H = -i\alpha \cdot \bigtriangledown + V, \ H_s = -i\alpha \cdot \bigtriangledown + V$$

The first one – on $\left[L^2(\Omega)\right]^4$ with boundary conditions $f_3 = f_4 = 0$ on Σ. The second operator – on $\left[L^2(\mathbb{R}^3)\right]^4$. Potential $V(x)$ is always assumed to be symmetric 4×4 matrix with the uniformly bounded entries. For H, it is given on Ω, for H_s – on $\mathbb{R}^3$. Thus, we have two self-adjoint operators $\mathcal{H}$ and $\mathcal{H}_s$. We also assume that after the "spherical change of variables"[2], the matrix of V has the canonical form in a sense of section 2. The typical example of such a potential is $V(x) = v(x)\beta$, where $v(x)$ is a scalar real-valued function (see [20], p.108 for other interactions and physical explanations). For simplicity, we will deal with this type of potentials only.

For α, β, we have

$$\alpha_k\alpha_l + \alpha_l\alpha_k = 2\delta_{kl}, \ \alpha_k\beta + \beta\alpha_k = 0, \ \beta^2 = 1; \ k, l = 1, 2, 3$$

Notice that by letting $I = -i\sigma_1, J = -i\sigma_2, K = -i\sigma_3$, we have relations

$$I^2 = J^2 = \lambda^2 = -1,$$

$$IJ = -JI, \ IK = -KI, \ KJ = -JK, \ IJ = K, \ JK = I, \ KI = J$$

which makes I, J, and K quaternions. The following algebraic relations are easy to verify.

Lemma 3.1. *If $\gamma \in \Sigma$, then*

$$\begin{aligned}
&(\alpha_1\gamma_1 + \alpha_2\gamma_2 + \alpha_3\gamma_3)^2 = 1 \\
&(\alpha_1\gamma_1 + \alpha_2\gamma_2 + \alpha_3\gamma_3 + 1)^2 = 2(\alpha_1\gamma_1 + \alpha_2\gamma_2 + \alpha_3\gamma_3 + 1) \\
&(\alpha_1\gamma_1 + \alpha_2\gamma_2 + \alpha_3\gamma_3 + 1)\beta(\alpha_1\gamma_1 + \alpha_2\gamma_2 + \alpha_3\gamma_3 + 1) = 0
\end{aligned}$$

Lemma 3.2. *The resolvent kernel of the free Dirac operator $\mathcal{H}_s^0$ has the following form (*[20]*, p. 39)*

$$G_\lambda^0(x, s) = \left(i\frac{\alpha \cdot (x - s)}{|x - s|^2} + \lambda\frac{\alpha \cdot (x - s)}{|x - s|} + \lambda \right) \frac{e^{i\lambda|x-s|}}{4\pi|x - s|} \tag{39}$$

---

[1]This rigidity is also present for polynomials orthogonal on the unit circle.
[2]We will define this change of variables later.

Proof. Use the identities

$$(\mathcal{H}_s^0-\lambda)^{-1}=(\mathcal{H}_s^0+\lambda)({\mathcal{H}_s^0}^2-\lambda^2)^{-1},\ {\mathcal{H}_s^0}^2=-\Delta \tag{40}$$

and the explicit formula for the resolvent kernel of the free Laplacian.

We will start with the following theorem.

Theorem 3.1. *Assume $V(x)=v(x)\beta$, where $v(x)$ is a real-valued, uniformly bounded, scalar function satisfying the following condition*

$$\int\limits_{x\in\Omega}\frac{v^2(x)}{|x|^2+1}\leq\infty \tag{41}$$

Then, $\sigma_{ac}(\mathcal{H})=\mathbb{R}$.

Proof. As in [10], we can assume without loss of generality, that $v(x)=0$ in $1<|x|<2$. Indeed, otherwise we subtract function $v(x)\beta\chi_{1<|x|<2}(x)$ from $v(x)\beta$. The resolvent of $\mathcal{H}^0$ is an integral operator. One can obtain an expression for its kernel by using identity (40) with $\mathcal{H}^0$ instead of $\mathcal{H}_s^0$. Therefore, the resolvent of $\mathcal{H}$ is an integral operator too and the trace-class argument [13] can be applied. Now, we take any nontrivial infinitely smooth radially-symmetric function $f(x)$ with support in $\{1<|x|<2\}$. We will show that the spectral measure of the element

$$f(x)=(f(r),0,0,0)^t \tag{42}$$

has an a.c. component which support fills $\mathbb{R}$. The proof consists of two steps. In the first step, we represent operator $\mathcal{H}$ in the canonical form (3) with unbounded operator-valued coefficients. Then, we apply the results of section 1 to obtain the needed estimates for the entropy of the spectral measure of f.

Let us start with the suitable change of variables. Consider the Dirac operator $\mathcal{H}$. Let us write this operator in the spherical coordinates (see [20], p.126 or [21], p.16). The standard unitary operator $f(x)\in L^2(\Omega)\xrightarrow{\mathcal{U}}F(r)=rf(r\sigma),\sigma\in\Sigma$ maps scalar functions of three variables to a vector-valued function of one variable. For almost any $r>1$, $F(r)\in L^2(\Sigma)$. Following [21], we consider the following auxiliary operators

$$\sigma_0=\left[\begin{array}{cc}1&0\\0&1\end{array}\right],\ \sigma_r=r^{-1}\sum_{j=1}^3\sigma_jx_j,\ p_j=-i\partial/\partial x_j,$$

$$p_r=-i(\partial/\partial r+r^{-1}),\ L_j=x_{j+1}p_{j+2}-x_{j+2}p_{j+1},\ s=\sigma_0+\sum_{j=1}^3\sigma_jL_j$$

where indices j are understood modulo 3, as usual. Notice that the operator p_r is unitary to $-id/dr$ under $\mathcal{U}$ and s is independent of r. We want to get a matrix representation of $\mathcal{H}$ convenient for us. To do that, we take a unitary, independent or r matrix U

$$U=\left[\begin{array}{cc}\sigma_0&0\\0&-i\sigma_r\end{array}\right]$$

Then, ([21], p. 18)

$$U^*\mathcal{H}U=\left[\begin{array}{cc}0&-i\sigma_0\\i\sigma_0&0\end{array}\right]p_r-\frac{1}{r}\left[\begin{array}{cc}0&s\\s&0\end{array}\right]+v(x)U^*\beta U \tag{43}$$

Operator s is selfadjoint in $[L^2(\Sigma)]^2$. Denote its orthonormal eigenfunctions by $\psi_{(n)}$ where n is a multiindex. Therefore, in $\left[L^2(\Sigma)\right]^4$, we can introduce an orthonormal basis spanned by the functions

$$\Psi^+_{(n)} = \begin{bmatrix} \psi_{(n)} \\ 0 \end{bmatrix}, \Psi^-_{(k)} = \begin{bmatrix} 0 \\ \psi_{(k)} \end{bmatrix}$$

Then, any function $f(x) \in \left[L^2(\Omega)\right]^4$ can be represented as a sum $f(x) \overset{\mathcal{U}}{\sim} F(r) = \sum\limits_n \varphi^+_{(n)}(r)\Psi^+_{(n)} + \sum\limits_k \varphi^-_{(k)}(r)\Psi^-_{(k)}$. The matrix of the unperturbed operator $\mathcal{H}_0$ can be written as ([20], p.128 and [21], p.22)

$$\begin{bmatrix} 0 & 0 & \dots & -\frac{d}{dr} - \frac{\kappa_{(1)}}{r} & 0 & \dots \\ 0 & 0 & \dots & 0 & -\frac{d}{dr} - \frac{\kappa_{(2)}}{r} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{d}{dr} - \frac{\kappa_{(1)}}{r} & 0 & \dots & 0 & 0 & \dots \\ 0 & \frac{d}{dr} - \frac{\kappa_{(2)}}{r} & \dots & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix} \begin{bmatrix} \varphi^+_{(1)}(r) \\ \varphi^+_{(2)}(r) \\ \dots \\ \varphi^-_{(1)}(r) \\ \varphi^-_{(2)}(r) \\ \dots \end{bmatrix}$$

where $\kappa_{(n)}$ are eigenvalues of s. The boundary conditions at $r = 1$ are $\varphi^-_{(n)}(1) = 0$.

Now, we want to work with the new Hilbert space $\mathcal{L}$, the L^2 space of vector-functions $\Phi(r) = (\varphi^+_{(1)}(r), \dots, \varphi^-_{(1)}(r), \dots)^t$, with the norm given by the formula below

$$\|\Phi(r)\|^2 = \int\limits_1^\infty \sum_n \left[|\varphi^+_{(n)}(r)|^2 + |\varphi^-_{(n)}(r)|^2\right] dr < \infty$$

So, the free multidimensional Dirac operator can be represented as an infinite orthogonal sum of one-dimensional Dirac operators. Using elementary properties of the Pauli matrices, we get

$$v(x)U^*\beta U = v(x)\beta$$

Consider the multiplication by $v(x)$ in $[L^2(\Sigma)]^2$ as a self-adjoint bounded operator with matrix $-b(r)$, $b(r) = \{b_{ij}(r)\}, i, j = 1, \dots, \infty$. Since the section 2 deals with an interval $[0, \infty)$ rather than $[1, \infty)$, we shift the argument r by one. We end up with the canonical representation (3), where $a(r)$ is an unbounded operator in the diagonal form for each $r > 0$. Notice that the constant function $(1, 0)^t \in [L^2(\Sigma)]^2$ is an eigenfunction of the operator s corresponding to the eigenvalue 1. For $\Psi^+_{(2)}$ and $\Psi^-_{(1)}$, we choose the normed vectors collinear to $(1, 0, 0, 0)^t$ and $(0, 0, 1, 0)^t$, respectively. We have $\kappa_{(1)} = 1$ and the function f, given by (42), corresponds to $\Phi(r) = ((r + 1)f(r + 1), 0, \dots, 0, \dots)$ in the vector-valued representation. From now on, we will deal with the representation (3) of an operator $\mathcal{H}$.

In the second step of the proof, we implement the main result of the second section, theorem 2.2. We can not do that directly, because we are dealing with an operator-valued coefficient $a(r)$ unbounded for each $r > 0$. To avoid this difficulty, we consider operators $\mathcal{D}_n$ with $b_n = P_n b P_n$, where P_n denotes the orthogonal projection onto the linear combination of the first n functions $\psi_{(k)}$. The matrix of

$b_n(r)$ is

$$b_n(r) = \begin{bmatrix} b_{11}(r) & \cdots & b_{1n}(r) & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n1}(r) & \cdots & b_{nn}(r) & 0 & \cdots \\ 0 & \cdots & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}, r > 0$$

Each of $\mathcal{D}_n$ can be written as an orthogonal sum of the Dirac operator $\hat{\mathcal{D}}_n$ with matrix-valued potential V_n of size $n\times n$ and an infinite number of the one-dimensional Dirac operators with scalar potentials. Denote by $\mathcal{L}_n$ the subspace of $\mathcal{L}$ on which $\hat{\mathcal{D}}_n$ acts. Matrix V_n has the following form

$$V_n = \begin{bmatrix} -b_n(r) & -a_n(r) \\ -a_n(r) & b_n(r) \end{bmatrix}, r > 0$$

with

$$a_n(r) = \begin{bmatrix} \frac{\kappa_{(1)}}{r+1} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \frac{\kappa_{(n)}}{r+1} \end{bmatrix}$$

Notice that the function $\Phi(r)$ lies in $\mathcal{L}_n$ for any n. Denote the spectral matrix-valued measure of $\hat{\mathcal{D}}_n$ by $\sigma_n(\lambda)$. Since $b(r) = 0$ on $[0, 1]$, the spectral measure $d\mu_n(\lambda)$ of $\Phi(r)$ is equal to $|\rho(\lambda)|^2(d\sigma_n(\lambda)e_1, e_1)$, where

$$\rho(\lambda) = \int_0^1 (r+1)f(r+1)\left[\cos(\lambda r) + \frac{\sin(\lambda r)}{\lambda}\right] dr$$

Here,

$$\cos(\lambda r) + \frac{\sin(\lambda r)}{\lambda}, \quad \frac{r\cos(\lambda r)}{\lambda(r+1)} - \sin(\lambda r)\left(1 + \frac{1}{\lambda^2(r+1)}\right)$$

are generalized eigenfunctions for the Dirac operator (3) with potential

$$V = \begin{bmatrix} 0 & -\dfrac{1}{r+1} \\ -\dfrac{1}{r+1} & 0 \end{bmatrix}$$

From the proof of theorem 2.2 (estimate (37)), we get

$$\int_\Delta \ln(\sigma_n'(\lambda)e_1, e_1)d\lambda \geq C_1 + C_2 \int_0^\infty \|(b_n(r) + ia_n(r))e_1\|^2 dr \tag{44}$$

for any finite interval $\Delta \in \mathbb{R}$. Notice that

$$\int_0^\infty \|a_n(r)e_1\|^2 dr = \int_0^\infty (r+1)^{-2} dr$$

and

$$\|b_n(r)e_1\|^2 \leq \|b(r)e_1\|^2 = \int_{\tau\in\Sigma} v^2((r+1)\tau)d\tau$$

So,

$$\int_0^\infty \|b_n(r)e_1\|^2 dr \le \int_0^\infty \int_{\tau\in\Sigma} v^2((r+1)\tau)d\tau dr \le C\int_\Omega \frac{v^2(x)}{|x|^2+1}dx$$

and we clearly have

$$\int_\Delta \ln \mu_n'(\lambda)d\lambda \ge C > -\infty \tag{45}$$

with some constant C independent of n. Notice that $b_n \to b$ strongly. So, $d\mu_n(\lambda)$ converges weakly to $d\mu$, the spectral measure of f with respect to the initial operator $\mathcal{H}$. The semicontinuity of the entropy [8] implies

$$\int_\Delta \ln \mu'(\lambda)d\lambda > C > -\infty$$

■

The reduction of $\mathcal{H}$ to a one-dimensional system with the operator-valued potential is, probably, not necessary. One could have introduced the radiative operator and worked with this operator directly avoiding an approximation by Dirac operators with matrix-valued potentials (see [16]). In the meantime, this reduction is not too difficult and it shows how very general facts for the matrix-valued orthogonal systems are applied to different PDE's.

We do not consider the question of existence of wave operators. It might be that the problem can be solved by using an approach of [7].

In the next theorem, we establish an asymptotics of the Green's function for the operator $\mathcal{H}_s$. For $\mathcal{H}$, that means existence of the function h (see (29)) satisfying homogeneous equation $Hh = \lambda h, \lambda \in \mathbb{C}^+$ with the well-controlled "amplitude". Due to (30) and (31), we have a certain estimate on the entropy of the corresponding spectral measure. That, in particular, gives another proof of $\sigma_{ac}(\mathcal{H}) = \mathbb{R}$ for the case of a power-decay.

Theorem 3.2. *Assume $v(x)$ is given on $\mathbb{R}^3$ and satisfies an estimate $|v(x)| < C_v(|x|+1)^{-0.5-\varepsilon}$, with fixed $\varepsilon > 0$ and C_v sufficiently small. Then, the resolvent kernel $G_\lambda(x,0)$ of operator H_s at a point $\lambda = i$ has the following representation*

$$G_i(x,0) = \frac{e^{-|x|}}{4\pi|x|}\left[\left(i\frac{\alpha\cdot x}{|x|}+i\right)\mathcal{P}_1(x) + (|x|+1)^{-0.5}\mathcal{P}_2(x)\right] \tag{46}$$

where

$$|\mathcal{P}_2(x)| < C \tag{47}$$

uniformly in $\mathbb{R}^3$, $\|\mathcal{P}_1 - 1\| < \delta$, $\delta \to 0$ as $C_v \to 0$. Positive constant C depends on C_v and ε only.

We need the smallness of C_v to guarantee convergence of a certain series. In general situation, one can take the spectral parameter λ sufficiently far from the spectrum. Or, we can take $\lambda = i$, and let potential be zero on the large ball centered at origin. That would make constant C_v as small as we want[3] but would not change the scattering picture.

Let us prove some auxiliary lemmas first.

[3] With respect to a smaller ε.

Lemma 3.3. *The following estimate holds* $(1 < \rho \le 2|x|/3)$

$$\int\limits_{|y|=\rho} e^{-|x-y|-|y|} d\tau_y < C\rho e^{-|x|} \tag{48}$$

$$\int\limits_{|y|=\rho} e^{-|x-y|-|y|} \sin\zeta(x,y) d\tau_y < C\sqrt{\rho} e^{-|x|}$$

$$\int\limits_{|y|=\rho} e^{-|x-y|-|y|} \sin^2\zeta(x,y) d\tau_y < C e^{-|x|}$$

where $\zeta(x,y)$ *is an angle between* x *and* y.

Proof. Without loss of generality, assume that $x = (0,0,|x|)$. Introducing the spherical coordinates $y_1 = \rho\cos\theta\cos\varphi, y_2 = \rho\cos\theta\sin\varphi, y_3 = \rho\sin\theta$, we get

$$\rho^2 \int\limits_{-\pi}^{\pi} d\varphi \int\limits_{-\pi/2}^{\pi/2} d\theta \cos\theta \exp(-\rho - \sqrt{|x|^2+\rho^2-2|x|\rho\sin\theta})$$

$$< \quad C\rho^2 e^{-|x|} \int\limits_{-\pi/2}^{\pi/2} d\theta\cos\theta\exp\left[-c|x|\rho(|x|-\rho)^{-1}(1-\sin\theta)\right] < Ce^{-|x|}\frac{|x|-\rho}{|x|}\rho$$

Estimate (48) is now straightforward. The other statement of the lemma can be proved similarly.■

Take any two vectors $x, y \in \mathbb{R}^3$. The following inequality is obvious

$$\left|\frac{y}{|y|} - \frac{x}{|x|}\right| < C\sin\zeta \tag{49}$$

Together with $\zeta(x,y)$, consider $\chi(x,y)$ – an angle between $x-y$ and x. From the sine-theorem, $\sin\chi = |y-x|^{-1}|y|\sin\zeta$. Consequently,

$$\left|\frac{x-y}{|x-y|} - \frac{x}{|x|}\right| < C\sin\chi = C|y-x|^{-1}|y|\sin\zeta \tag{50}$$

By the triangle inequality,

$$\left|\frac{y}{|y|} - \frac{x-y}{|x-y|}\right| \le \left|\frac{y}{|y|} - \frac{x}{|x|}\right| + \left|\frac{x}{|x|} - \frac{x-y}{|x-y|}\right| \tag{51}$$

For $|y| < 2|x|/3$,

$$\left|\frac{y}{|y|} - \frac{x-y}{|x-y|}\right| \le C\sin\zeta \tag{52}$$

Let $|x| > 1$ and $\Upsilon = \{y : |y| > 2|x|/3, |x-y| > 2|x|/3\}$.

Lemma 3.4. *The following estimate holds*

$$\int\limits_{\Upsilon} \exp(-|x-y|-|y|)dy \le C\exp(-\gamma|x|) \tag{53}$$

with $\gamma > 1$.

Proof. Indeed, in Υ,

$$|x-y|+|y|>|y|/5+16|x|/15$$

Taking $\gamma=16/15$, we obtain the statement of the lemma. ■

In the following three lemmas, we will be estimating certain integrals over the $\mathbb{R}^3$. Lemma 3.4 shows that the contribution coming from the integration over Υ is small and can be neglected.

Lemma 3.5. *The following bound is true*

$$\int_{\mathbb{R}^3} \frac{e^{-|x-y|}}{|x-y|^2} \frac{e^{-|y|}}{|y|^{1.5+\varepsilon}+1} dy < C \frac{e^{-|x|}}{|x|^{1.5}+1}$$

Proof. By lemma 3.3,

$$\int_{|y|<2|x|/3} \frac{e^{-|x-y|}}{|x-y|^2} \frac{e^{-|y|}}{|y|^{1.5+\varepsilon}+1} dy < C \frac{e^{-|x|}}{|x|^2+1} \int_0^{|x|} \frac{\rho}{\rho^{1.5+\varepsilon}+1} d\rho < C \frac{e^{-|x|}}{|x|^{1.5+\varepsilon}+1};$$

$$\int_{|y-x|<2|x|/3} \frac{e^{-|x-y|}}{|x-y|^2} \frac{e^{-|y|}}{|y|^{1.5+\varepsilon}+1} dy = \int_{|y|<2|x|/3} \frac{e^{-|x-y|}}{|y|^2} \frac{e^{-|y|}}{|x-y|^{1.5+\varepsilon}+1} dy$$

$$< C \frac{e^{-|x|}}{|x|^{1.5+\varepsilon}+1} \int_0^{|x|} \frac{d\rho}{\rho+1} < C \frac{e^{-|x|}}{|x|^{1.5}+1}$$

■

Lemma 3.6. *The following relation is true*

$$\begin{aligned} &\int_{\mathbb{R}^3} \frac{e^{-|x-y|}}{|x-y|} \left(\frac{\alpha\cdot(x-y)}{|x-y|}+1\right) \frac{e^{-|y|}}{|y|^{2+\varepsilon}+1} dy \\ = \quad & \frac{e^{-|x|}}{|x|+1} \left[\left(\frac{\alpha\cdot x}{|x|}+1\right)\varphi_1(x) + (|x|+1)^{-0.5}\varphi_2(x)\right] \end{aligned}$$

where $\varphi_{1(2)}(x)$ are matrix-functions uniformly bounded in $\mathbb{R}^3$.

Proof.

$$\begin{aligned} &\int_{\mathbb{R}^3} \frac{e^{-|x-y|}}{|x-y|} \left(\frac{\alpha\cdot(x-y)}{|x-y|}+1\right) \frac{e^{-|y|}}{|y|^{2+\varepsilon}+1} dy \\ = &\int_{\mathbb{R}^3} \frac{e^{-|x-y|}}{|x-y|} \left(\frac{\alpha\cdot(x-y)}{|x-y|} - \frac{\alpha\cdot x}{|x|}\right) \frac{e^{-|y|}}{|y|^{2+\varepsilon}+1} dy \\ &+ \left(\frac{\alpha\cdot x}{|x|}+1\right) \int_{\mathbb{R}^3} \frac{e^{-|x-y|}}{|x-y|} \frac{e^{-|y|}}{|y|^{2+\varepsilon}+1} dy \quad = \quad I_1+I_2 \end{aligned}$$

Following the proof of lemma 3.5, we get

$$\int_{\mathbb{R}^3} \frac{e^{-|x-y|}}{|x-y|} \frac{e^{-|y|}}{|y|^{2+\varepsilon}+1} dy < C \frac{e^{-|x|}}{|x|+1} \quad \text{(sharp!)}$$

For I_1, we have

$$\int_{|x-y|<2|x|/3} \frac{e^{-|x-y|}}{|x-y|}\left(\frac{\alpha\cdot(x-y)}{|x-y|}-\frac{\alpha\cdot x}{|x|}\right)\frac{e^{-|y|}}{|y|^{2+\varepsilon}+1}dy$$
$$= \quad -\int_{|y|<2|x|/3} \frac{e^{-|y|}}{|y|}\left(\frac{\alpha\cdot y}{|y|}-\frac{\alpha\cdot x}{|x|}\right)\frac{e^{-|x-y|}}{|x-y|^{2+\varepsilon}+1}dy$$

By (49) and lemma 3.3,

$$\int_{|y|<2|x|/3} \frac{e^{-|y|}}{|y|}\left|\frac{y}{|y|}-\frac{x}{|x|}\right|\frac{e^{-|x-y|}}{|x-y|^{2+\varepsilon}+1}dy < \frac{Ce^{-|x|}}{|x|^{2+\varepsilon}+1}\int_0^{|x|}\rho^{-0.5}d\rho < C\frac{e^{-|x|}}{|x|^{1.5+\varepsilon}+1}$$

We now estimate integral in I_1 over $|y|<2|x|/3$. By (50) and lemma 3.3,

$$\left|\int_{|y|<2|x|/3} \frac{e^{-|x-y|}}{|x-y|}\left(\frac{\alpha\cdot(x-y)}{|x-y|}-\frac{\alpha\cdot x}{|x|}\right)\frac{e^{-|y|}}{|y|^{2+\varepsilon}+1}dy\right|$$
$$< \quad \frac{Ce^{-|x|}}{|x|^2+1}\int_0^{|x|}\frac{\rho^{1.5}}{\rho^{2+\varepsilon}+1}d\rho < C\frac{e^{-|x|}}{|x|^{1.5+\varepsilon}+1}$$

■

Lemma 3.7. *If $v(x)$ is a real-valued scalar function satisfying an estimate $|v(x)|<(|x|+1)^{-0.5-\varepsilon}$, then the following representation holds*

$$\int_{\mathbb{R}^3} \frac{e^{-|x-y|}}{|x-y|}\left(\frac{\alpha\cdot(x-y)}{|x-y|}+1\right)\beta v(y)\left(\frac{\alpha\cdot y}{|y|}+1\right)\frac{e^{-|y|}}{|y|}dy$$
$$= \quad \frac{e^{-|x|}}{|x|+1}\left[\left(\frac{\alpha\cdot x}{|x|}+1\right)\varphi_1(x)+(|x|+1)^{-0.5}\varphi_2(x)\right]$$

where $\varphi_{1(2)}(x)$ are, again, matrix-functions uniformly bounded in $\mathbb{R}^3$.

Proof. From the lemma 3.1, we infer

$$\left(\frac{\alpha\cdot(x-y)}{|x-y|}+1\right)\beta\left(\frac{\alpha\cdot(x-y)}{|x-y|}+1\right)=0$$

Therefore,

$$\int_{\mathbb{R}^3} \frac{e^{-|x-y|}}{|x-y|}\left(\frac{\alpha\cdot(x-y)}{|x-y|}+1\right)\beta v(y)\left(\frac{\alpha\cdot y}{|y|}+1\right)\frac{e^{-|y|}}{|y|}dy \quad =$$
$$= \int_{\mathbb{R}^3} \frac{e^{-|x-y|}}{|x-y|}\left(\frac{\alpha\cdot(x-y)}{|x-y|}+1\right)\beta v(y)\left(\frac{\alpha\cdot y}{|y|}-\frac{\alpha\cdot(x-y)}{|x-y|}\right)\frac{e^{-|y|}}{|y|}dy$$
$$= J_1+J_2$$

where

$$J_1=\left(\frac{\alpha\cdot x}{|x|}+1\right)\int_{\mathbb{R}^3}\frac{e^{-|x-y|}}{|x-y|}\beta v(y)\left(\frac{\alpha\cdot y}{|y|}-\frac{\alpha\cdot(x-y)}{|x-y|}\right)\frac{e^{-|y|}}{|y|}dy$$

and

$$J_2=\int_{\mathbb{R}^3}\frac{e^{-|x-y|}}{|x-y|}\left(\frac{\alpha\cdot(x-y)}{|x-y|}-\frac{\alpha\cdot x}{|x|}\right)\beta v(y)\left(\frac{\alpha\cdot y}{|y|}-\frac{\alpha\cdot(x-y)}{|x-y|}\right)\frac{e^{-|y|}}{|y|}dy$$

Consider J_1. Using (52) and lemma 3.3, we obtain

$$\int_{|y|<2|x|/3} \frac{e^{-|x-y|}}{|x-y|}|v(y)|\left|\frac{y}{|y|}-\frac{x-y}{|x-y|}\right|\frac{e^{-|y|}}{|y|}dy$$

$$< \quad C\frac{e^{-|x|}}{|x|+1}\int\limits_0^{|x|}\frac{\rho^{0,5}}{\rho^{1.5+\varepsilon}+1}d\rho < C\frac{e^{-|x|}}{|x|+1} \quad \text{(sharp!)}$$

$$\int_{|y-x|<2|x|/3} \frac{e^{-|x-y|}}{|x-y|}|v(y)|\left|\frac{y}{|y|}-\frac{x-y}{|x-y|}\right|\frac{e^{-|y|}}{|y|}dy$$

$$= \int_{|y|<2|x|/3} \frac{e^{-|x-y|}}{|x-y|}|v(x-y)|\left|\frac{y}{|y|}-\frac{x-y}{|x-y|}\right|\frac{e^{-|y|}}{|y|}dy$$

$$< C\frac{e^{-|x|}}{|x|^{1.5+\varepsilon}+1}\int\limits_0^{|x|}\frac{\rho^{0.5}}{\rho+1}d\rho < C\frac{e^{-|x|}}{|x|^{1+\varepsilon}+1}$$

Using (50), (52), and lemma 3.3, we get the following inequalities to estimate $|J_2|$

$$|J_2| \leq \int_{|y|<2|x|/3} \frac{e^{-|x-y|}}{|x-y|}\left|\frac{x-y}{|x-y|}-\frac{x}{|x|}\right||v(y)|\left|\frac{y}{|y|}-\frac{x-y}{|x-y|}\right|\frac{e^{-|y|}}{|y|}dy$$

$$< \quad \frac{Ce^{-|x|}}{|x|^2+1}\int\limits_0^{|x|}\frac{\rho}{\rho^{1.5+\varepsilon}+1}d\rho < \frac{Ce^{-|x|}}{|x|^{1.5+\varepsilon}+1}$$

For the region $|x-y|<2|x|/3$, we have

$$\int_{|y-x|<2|x|/3} \frac{e^{-|x-y|}}{|x-y|}\left|\frac{x-y}{|x-y|}-\frac{x}{|x|}\right||v(y)|\left|\frac{y}{|y|}-\frac{x-y}{|x-y|}\right|\frac{e^{-|y|}}{|y|}dy \quad =$$

$$\int_{|y|<2|x|/3} \frac{e^{-|x-y|}}{|x-y|}\left|\frac{y}{|y|}-\frac{x}{|x|}\right||v(x-y)|\left|\frac{y}{|y|}-\frac{x-y}{|x-y|}\right|\frac{e^{-|y|}}{|y|}dy$$

$$< \frac{Ce^{-|x|}}{|x|^{1.5+\varepsilon}+1}\int\limits_0^{|x|}\frac{d\rho}{\rho+1} < \frac{Ce^{-|x|}}{|x|^{1.5}+1}$$

■

Proof of the theorem 3.2. Let us iterate the second resolvent identity to get the needed estimate for $G_i(x,0)$

$$G_i(x,0) = G_i^0(x,0) - \int\limits_{\mathbb{R}^3} G_i^0(x,s)\beta v(s)G_i(s,0)ds$$

Using lemmas 3.5–3.7 and explicit formula for $G_i^0(x,s)$, we see that the $n-$th term in the corresponding series has the following form

$$\frac{e^{-|x|}}{|x|+1}\left[\left(\frac{\alpha\cdot x}{|x|}+1\right)\varphi_1^{(n)}(x)+(|x|+1)^{-0.5}\varphi_2^{(n)}(x)\right] \tag{54}$$

with $|\varphi_{1(2)}^{(n)}(x)| < [C(\varepsilon)C_v]^n$. That can be easily proved by the induction. Indeed, for the first term, we have the representation (39). Assume that (54) holds for the

n-th term T_n. Then,

$$T_{n+1}(x) = -\int\limits_{\mathbb{R}^3} G_i^0(x,s)\beta v(s) T_n(s) ds = -(I_1 + I_2 + I_3 + I_4)$$

where

$$I_1(x) = \frac{i}{4\pi}\int\limits_{\mathbb{R}^3} \frac{\exp(-|x-s|)}{|x-s|}\left(\frac{\alpha\cdot(x-s)}{|x-s|}+1\right)\beta v(s)\left(\frac{\alpha\cdot s}{|s|}+1\right)\frac{\exp(-|s|)}{|s|+1}\varphi_1^{(n)}(s)ds$$

$$I_2(x) = \frac{i}{4\pi}\int\limits_{\mathbb{R}^3} \frac{\exp(-|x-s|)}{|x-s|}\left(\frac{\alpha\cdot(x-s)}{|x-s|}+1\right)\beta v(s)\left(\frac{\exp(-|s|)}{(|s|+1)^{1.5}}\right)\varphi_2^{(n)}(s)ds$$

$$I_3(x) = \frac{i}{4\pi}\int\limits_{\mathbb{R}^3} \frac{\alpha\cdot(x-s)}{|x-s|^2}\left(\frac{\exp(-|x-s|)}{|x-s|}\right)\beta v(s)\left(\frac{\alpha\cdot s}{|s|}+1\right)\frac{\exp(-|s|)}{|s|+1}\varphi_1^{(n)}(s)ds$$

$$I_4(x) = \frac{i}{4\pi}\int\limits_{\mathbb{R}^3} \frac{\alpha\cdot(x-s)}{|x-s|^2}\left(\frac{\exp(-|x-s|)}{|x-s|}\right)\beta v(s)\left(\frac{\exp(-|s|)}{(|s|+1)^{1.5}}\right)\varphi_2^{(n)}(s)ds$$

We apply lemma 3.7 to I_1, lemma 3.6 to I_2, and lemma 3.5 to I_3 and I_4. If C_v is small enough, we will get uniform convergence of the series and the estimate (47). Clearly, $\|\mathcal{P}_1 - 1\| < \delta$ with $\delta \to 0$ if $C_v \to 0$. ■

Remark. It is clear that the asymptotics of the Green's function $G_i(x,s), |s|<1$ can be obtained similarly: it is close to $G_i^0(x,s)$ as $|x|\to\infty$.

Fix any $\varepsilon > 0$. Consider the Dirac operator $\mathcal{H}$ with $|v(x)| < C_v(|x|+1)^{-0.5-\varepsilon}$, where C_v is sufficiently small, and operator $\mathcal{H}_s$ on $\mathbb{R}^3$ with $v(x)=0$ on $|x|<1$. Then, for any nonzero function $f(x) = (f_1,f_2,f_3,f_4)^t \in \left[L^2(\mathbb{R}^3)\right]^4$ with support in the unit ball, the function

$$h(x) = \int\limits_{\mathbb{R}^3} G_i(x,s) f(s) ds$$

satisfies equation $Hh = ih$ (h does not satisfy boundary condition on Σ unless $h = 0$ in Ω). Therefore, by the theorem 3.2 and by the remark above, we can control the "amplitude" of h. For $|x|$ large enough,

$$e^{|x|}|x|h(x) = \frac{i}{4\pi}\left(\frac{\alpha\cdot x}{|x|}+1\right)\int\limits_{\mathbb{R}^3}\exp\langle\frac{x}{|x|},s\rangle f(s)ds + \hat{f}(x)$$

and

$$\|\hat{f}(x)\| < \delta||f||_1$$

with $\delta\to 0$ as $C_v \to 0$. For the integrals, we have

$$\int\limits_{\mathbb{R}^3}\exp\langle\frac{x}{|x|},s\rangle|f_j(s)|ds > e^{-1}\int\limits_{\mathbb{R}^3}|f_j(s)|ds, (j=1,\ldots,4)$$

The "amplitude" is calculated as

$$\mathcal{A}(|x|^{-1}x) = \lim_{|x|\to\infty}|x|\exp(|x|-1)(h_3(x),h_4(x))^t$$

Take $f_1 = f_2 = 0$ and f_3, f_4 – arbitrary non-negative. Then, $\mathcal{A}$ is well-defined as an element of $[L^2(\Sigma)]^2$ for $v_n = v(x)\chi_{|x|<n}(x)$. Denote this sequence by $\mathcal{A}_n$.

Now, theorem 3.2 says that $\mathcal{A}_n$ does not go to zero. Using the reduction to a one-dimensional system and (30), one can show that the spectral measure of an element f has an a.c. component which fills $\mathbb{R}$.

It is likely, that more careful estimates might prove some analog of theorem 3.2 for v satisfying condition (2). One of the main ideas of the paper is as follows. Take the spectral parameter as far from the spectrum as we wish. If we manage to find at least one solution to homogeneous problem that satisfies certain asymptotics at infinity with nonzero "amplitude", then an a.c. spectrum fills the whole line. The main advantage is that we need to do some rather delicate analysis of, say, Green's function far from the spectrum, not on the real line. That is an interesting problem to study the Green's function for operator (1) when the potential is random with slow decay or with no decay at all (the Anderson model). The approach of theorem 3.1 does not work in this case. In the meantime, analysis done in [3] together with estimates used in the proof of theorem 3.2 seem to be relevant for the problem.

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